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**LETTER TO THE EDITOR**

# Hannay angle in an *LCR* circuit with time-dependent inductance, capacity and resistance

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**Abstract**
The Hannay angle in an *LCR* circuit with time-dependent inductance, capacitance and resistance is obtained. A simple scheme to measure this angle is also proposed.

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Geometric phases and their physical effects have attracted considerable attention since Berry discovered the quantum geometric phase (Berry phase) [1] and Hannay found its classical correspondent (Hannay angle) [2]. The concept of geometric phase has penetrated many areas of physics, with the theoretical investigations and experimental findings of Berry phases and Hannay angles in many physical systems [3]. Here we show, based on the similarity between mechanical and electromagnetic oscillations, that there exists a Hannay angle (classical geometric phase) in an *LCR* circuit, a simple and well-known classical electromagnetic oscillation system, when the inductance, capacity and resistance of the circuit change with time periodically and adiabatically.

First, let us establish the differential equation for an *LCR* circuit with time-dependent inductance, capacitance and resistance, denoted by $L(t)$, $C(t)$ and $R(t)$, respectively. Here we assume that $L(t), C(t)$ and $R(t)$ satisfy the condition $4LC > (CR)^2$. If $I(t)$ is the current at time $t$ in the circuit, the potential difference of the resistor is $U_R = I(t)R(t)$ and the magnetic flux through the inductor is $\Phi = L(t)I(t)$. According to Faraday's law, the induced emf of the inductor is given by $\varepsilon_L = -\mathrm{d}\Phi/\mathrm{d}t$. If $Q(t)$ is the charge at time $t$ on the capacity plate, the potential difference of the capacity is $U_C = Q(t)/C(t)$. Since $I(t) = \mathrm{d}Q/\mathrm{d}t$, applying Kirchhoff's law, one can get the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\frac{\mathrm{d}Q}{\mathrm{d}t}\right) + R(t)\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C(t)} = 0. \tag{1}$$

This equation can be recognized as the same differential equation for a time-dependent damped harmonic oscillator by making use of the correspondence relationship between the physical quantities of mechanical oscillators and those of electric circuits, which is as follows:

displacement $x \Leftrightarrow$ electric charge $Q$

mass $m \Leftrightarrow$ inductance $L$

damping coefficient $\mu \Leftrightarrow$ resistance $R$

spring coefficient $k \Leftrightarrow$ inverse of capacitance $1/C$.

With the help of this well-known relationship, one can find that the differential equation for the corresponding mechanical oscillator is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\frac{\mathrm{d}x}{\mathrm{d}t}\right) + \mu(t)\frac{\mathrm{d}x}{\mathrm{d}t} + k(t)x = 0. \tag{2}$$

This is just Newton's equation for a damped harmonic oscillator with time-dependent mass, spring coefficient and damping coefficient.

In the Hamiltonian formulation of classical mechanics, the Hamiltonian for a time-dependent damped harmonic oscillator is [4]

$$H = \exp\left(-\int_0^t \frac{\mu}{m}\,\mathrm{d}t\right)\frac{p^2}{2m} + \exp\left(\int_0^t \frac{\mu}{m}\,\mathrm{d}t\right)\frac{kx^2}{2}. \tag{3}$$

If we perform a canonical transformation given by

$$x' = x\exp\left(\int_0^t \frac{\mu}{2m}\,\mathrm{d}t\right) \qquad p' = p\exp\left(-\int_0^t \frac{\mu}{2m}\,\mathrm{d}t\right) \tag{4}$$

the new Hamiltonian turns out to be

$$H' = \tfrac{1}{2}[X(t)x'^2 + 2Y(t)p'x' + Z(t)p'^2] \tag{5}$$

where $X(t) = k(t), Y(t) = \mu(t)/2m(t), Z(t) = 1/m(t)$, satisfying the condition $XZ - Y^2 > 0$. Hamiltonian (5) is just the Hamiltonian for a generalized time-dependent harmonic oscillator that was studied by Berry in [5]. In this paper, Berry showed that if the parameters $X, Y, Z$ change with time periodically and adiabatically, there exists a classical geometric phase (Hannay angle), besides a dynamical phase $\varphi_D = \int_0^T \omega_D\,\mathrm{d}t$, in the total phase of the $x'(t)$. The Hannay angle acquired in one period is

$$\Delta\theta = \oint \mathrm{d}\vec{R}\cdot\frac{Z}{2\omega_D}\nabla_{\vec{R}}\left(\frac{Y}{Z}\right). \tag{6}$$

Here $\omega_D = \sqrt{XZ - Y^2}$; $\vec{R} = (X, Y, Z)$ denotes the vector in parameter space. It is obvious that the Hannay angle is only dependent on the path in parameter space.

Based on the similarity between mechanical and electromagnetic oscillations, we can conclude that if the inductance, capacity and resistance of an *LCR* circuit change with time periodically and adiabatically, there exists a Hannay angle, besides a dynamical phase $\varphi_D = \int_0^T \sqrt{1/LC - (R/2L)^2}\,\mathrm{d}t$, in the total phase of $Q(t)$. The Hannay angle acquired in one period can be directly obtained as

$$\Delta\theta = \oint \mathrm{d}\vec{S}\cdot\frac{1}{4\sqrt{L/C - (R/2)^2}}\nabla_{\vec{S}}R = \oint \frac{\mathrm{d}R}{4\sqrt{L/C - (R/2)^2}}. \tag{7}$$

Here $\vec{S} = (L, C, R)$, represents the vector in parameter space.

Since the realization of an *LCR* circuit with time-dependent parameters is easier than that of the corresponding mechanical oscillator, one may naturally expect that the experimental finding of the Hannay angle in the circuit should be easier. Now we turn to the issue of how to measure this angle in the experiment.

It is heuristic to study how dynamical phase and geometric phase change when we perform a scale transformation on the inductance and capacity. One can observe that if a scale transformation, given by $L(t) \to aL(t)$, $C(t) \to aC(t)$, is performed on the inductance and capacity, the dynamical phase will transform as $\varphi_D \to \varphi_D/a$, and the geometric phase will be unchanged. This fact leads to a simple and straightforward way to measure the Hannay angle in the circuit.

Consider two time-dependent *LCR* circuits, one with inductance $L(t)$, capacitance $C(t)$ and resistance $R(t)$, and the other with $2L(t)$, $2C(t)$, $R(t)$. Assuming that (1) the two circuits begin to oscillate at the same time and with the same initial phases, and (2) the inductances, capacities and resistances change periodically and adiabatically, one can find that after one period the dynamical phase in the second circuit is half of that in the first and the Hannay angles in both the circuits are the same. Let $\varphi_1$ and $\varphi_2$ denote the total phases of the charges on the capacity plates in the first and second circuits, respectively. The foregoing analysis suggests that if $\varphi_1$ and $\varphi_2$ are measured in the experiment, the Hannay angle in either circuit is given by

$$\Delta\theta = 2\varphi_2 - \varphi_1. \tag{8}$$

No matter how the adiabatic and periodic changes of inductances, capacitances and resistances in the circuits are governed, if the paths in parameter space are the same, $\Delta\theta$ will be unchanged, since the Hannay angle is only dependent on the path in parameter space.

In conclusion, the Hannay angle in an *LCR* circuit with time-dependent inductance, capacitance and resistance has been investigated in this letter. Based on the similarity between the mechanical and electromagnetic oscillations, we have shown that there exists a Hannay angle when the parameters of the circuit change with time periodically and adiabatically. A simple scheme to measure this angle has also been proposed. As a final remark, we expect experimental observations of the Hannay angle in the time-dependent *LCR* circuit.

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